Stochastic Integration for Chaotic Dynamical Systems

Emily Williams, David Darmofal

First-Year CSGF Class of 2022, Massachusetts Institute of Technology

Overview and Objectives

Stochastic processes are used often in mathematical modeling of phenomena that appear to vary chaotically or in a random manner. Stochastic differential equations are ubiquitous in the formulation of these models, including blood clotting, population dynamics, neuron activity, turbulent diffusion, and more. We aim to incorporate stochastic modeling to improve the time integration schemes for chaotic systems.

Stochastic Differential Equations + Multiscale Discretizations + Subgrid-Scale Modeling = **Better Stabilized Models for Chaotic Systems**

Initial efforts toward this investigation include developing a higher-order finite-element-based time-marching approach based on the discontinuous Galerkin (DG) scheme [1] for stochastic and chaotic initial value problems (IVPs):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0$$

Stochastic Initial Value Problems

Consider a stochastic IVP:

$$d\mathbf{x} = \mathbf{f}(\mathbf{x})dt + \mathbf{b}(\mathbf{x})dW, \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{f}(\mathbf{x}) = \alpha\mathbf{x}$ and $\mathbf{b}(\mathbf{x}) = \beta\mathbf{x}$. Solving for $\dot{\mathbf{x}}$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \frac{1}{2}\mathbf{b}(\mathbf{x})\frac{\partial \mathbf{b}(\mathbf{x})}{\partial \mathbf{x}} + \mathbf{b}(\mathbf{x})dW/dt$$

Fig. 1 shows results for $\alpha = 2, \beta = 0.5$, and $x_0 = 1$.

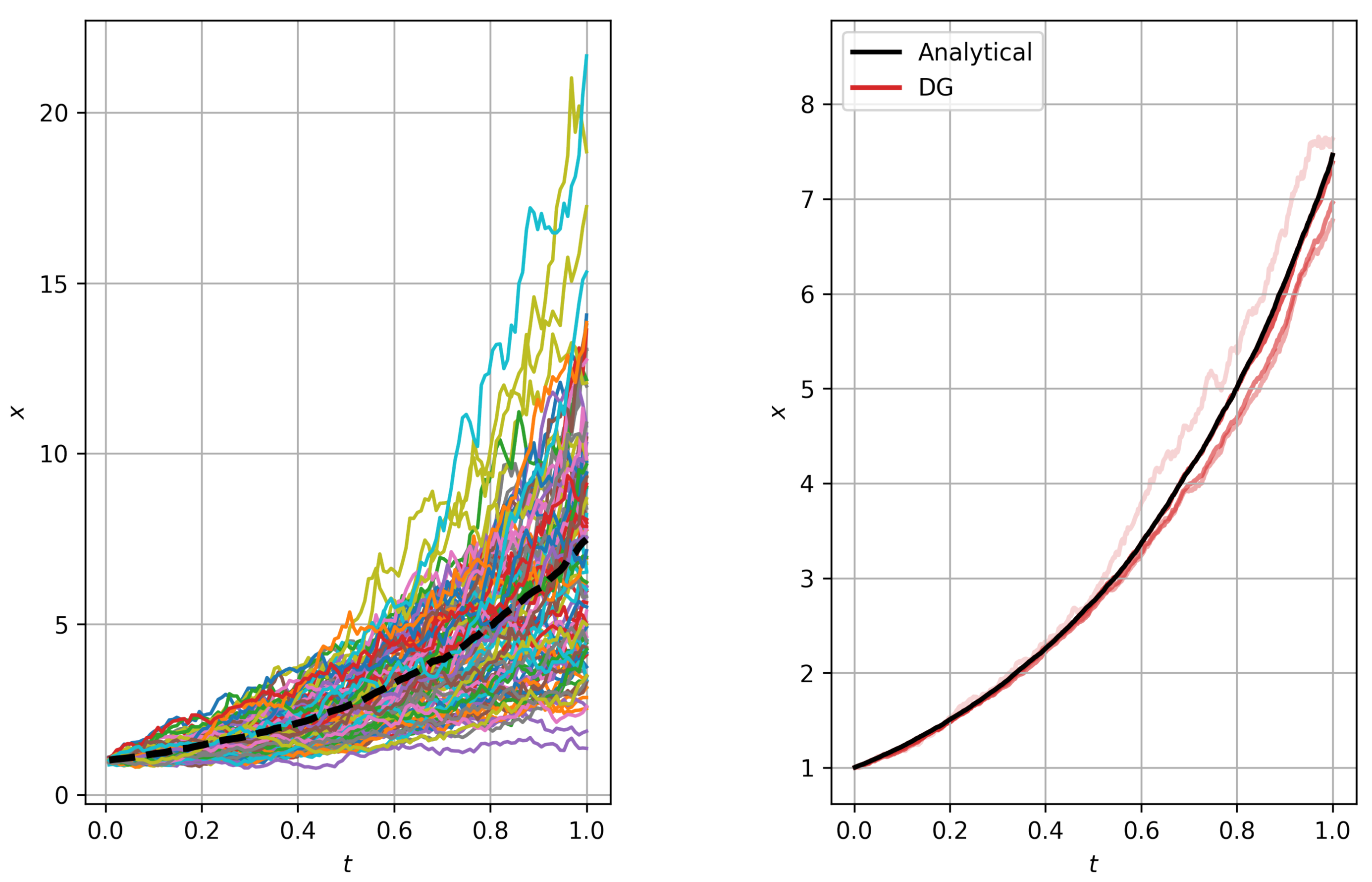

Fig. 1: Exact realizations of analytical solution and expectation of approximation

Chaotic Dynamical Systems

Consider the Lorenz system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} = (x, y, z)$ and:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \sigma(y-x) \\ x(r-z)-y \\ xy - \beta z \end{bmatrix}$$

with parameter values $\sigma = 10$, $r = 28$, and $\beta = 8/3$. We start at $\mathbf{x}_0$ and integrate over $t \in [-5, 15]$ with $\Delta t = 0.001$ and $p = 3$ using the DG scheme to acquire the trajectory of the Lorenz system shown in Fig. 2.

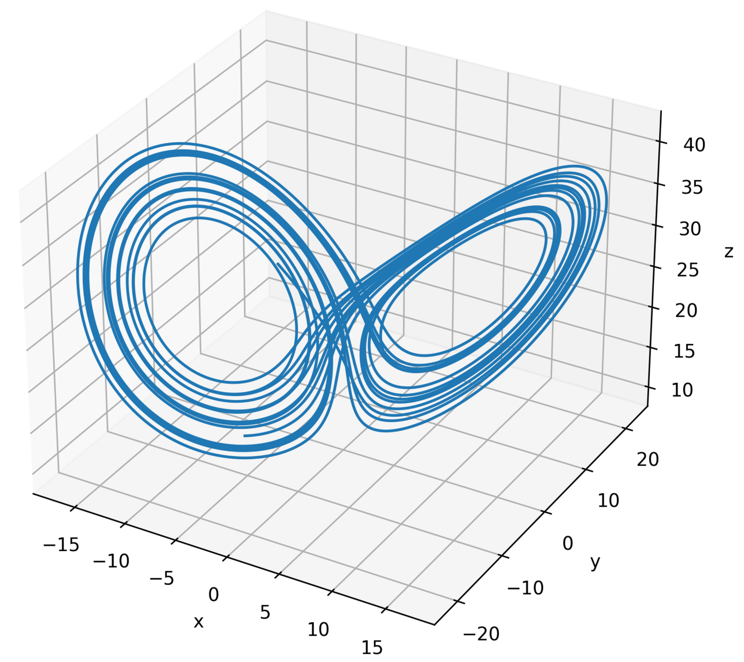

Fig. 2: Lorenz system

A spatial filter is defined by some filter width Δ and can be thought of as a moving average within each Δ in time [2].

$$\overline{\phi}(t) = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \phi(t+\tau)\, d\tau$$

Fig. 3 shows the filtered states of $\mathbf{x}$ and dynamics $\mathbf{f}(\mathbf{x})$ for $\Delta = 0.25$.

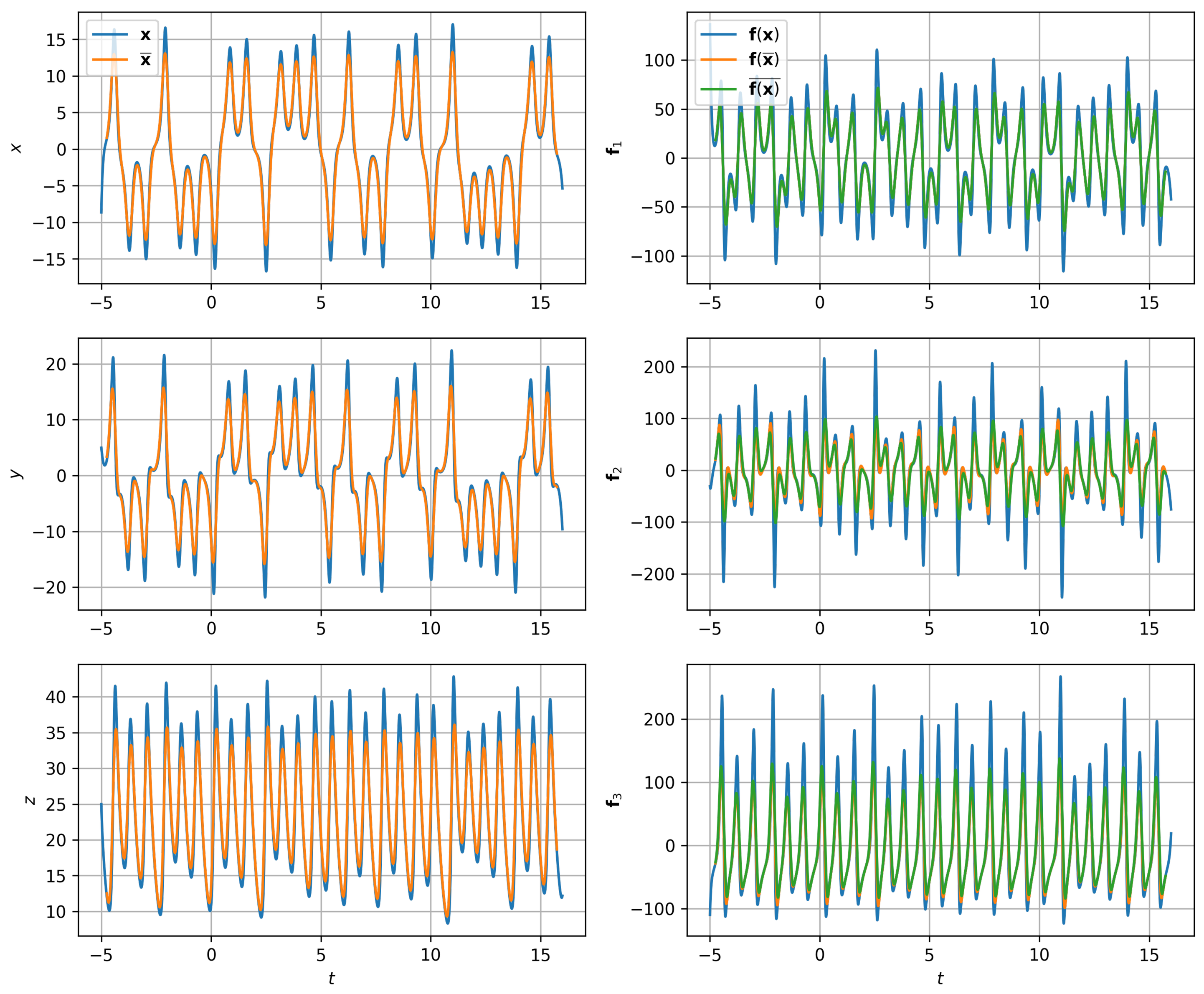

Fig. 3: Filtered states of Lorenz system

Subgrid-Scale Modeling

Many stochastic integration schemes use the Brownian increment $\Delta W \sim \sqrt{\Delta t}\mathcal{N}(0,1)$. We propose that this increment should be sampled from some function that is representative of the subgrid (i.e., unresolved) dynamics $\mathbf{s}$ of the chaotic system that emerge when filtering.

$$\dot{\overline{\mathbf{x}}} = \overline{\mathbf{f}(\mathbf{x})} = \mathbf{f}(\overline{\mathbf{x}}) + \mathbf{s}(\overline{\mathbf{x}}, \mathbf{x})$$
$$\mathbf{s}(\overline{\mathbf{x}}, \mathbf{x}) = \overline{\mathbf{f}(\mathbf{x})} - \mathbf{f}(\overline{\mathbf{x}})$$

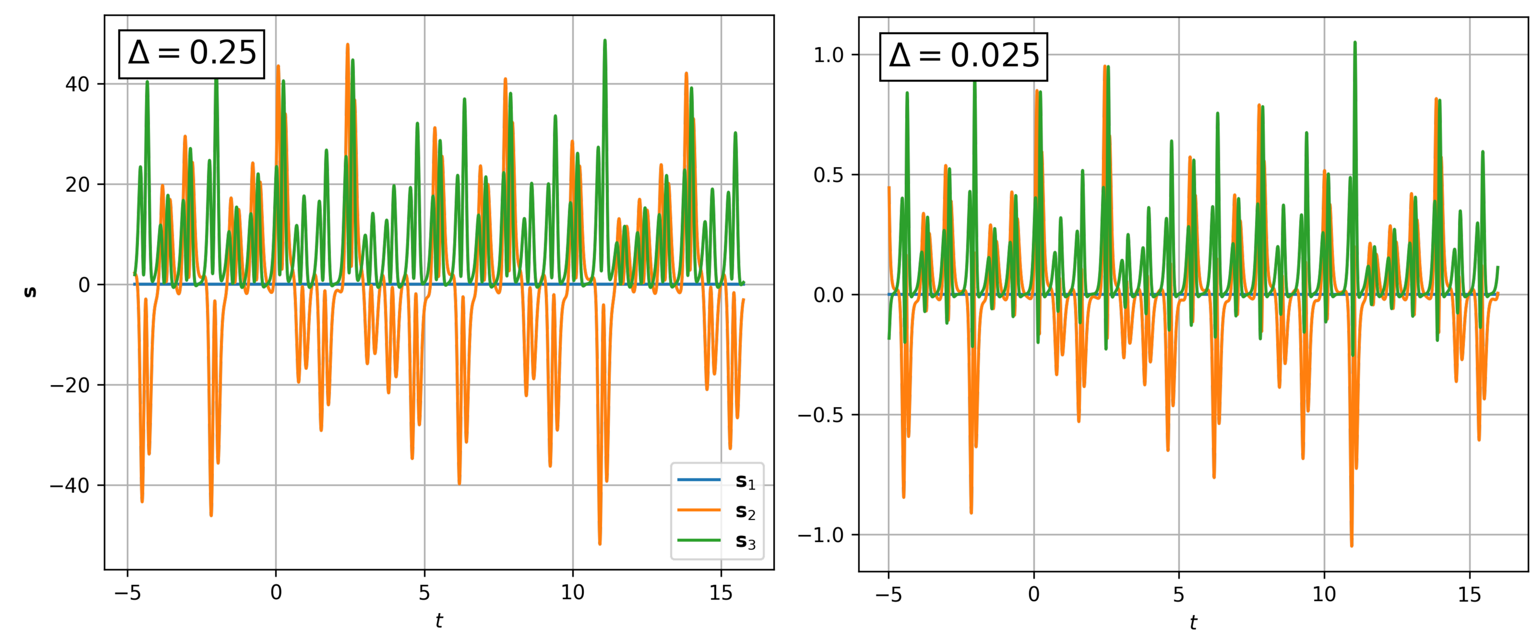

Fig. 4: Subgrid dynamics of Lorenz system

High-Performance Computing & Future Work

- Formulate stochastic subgrid modeling using a variational framework.
- Explore the potential stabilizing effect that stochastic models could have on linearized sensitivity of chaotic systems. Existing methods to compute the sensitivity derivatives for statistical quantities are costly [3].
- Efficient algorithms and HPC will both be useful when approximating large-scale systems, like computational fluid dynamics problems using large-eddy simulation (LES).

References

[1] W. H. Reed and T. R. Hill, “Triangular mesh methods for the neutron transport equation,” 1973.

[2] A. Leonard, “Energy cascade in large-eddy simulations of turbulent fluid flows,” *Advances in Geophysics*, 1975.

[3] Q. Wang, “Forward and adjoint sensitivity computation of chaotic dynamical systems,” *Journal of Computational Physics*, 2013.